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A SIMPLE EXAMPLE OF BLACK NOISE

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ABSTRACT. – By a *noise* in continuous time $t \in \mathbf{R}$, we mean a family $\{\mathcal{F}_{s,t},\ s \leqslant t\}$ of sub σ-fields of events on a separable probability space with properties given in Definition 1.2, a property of independence, in particular. A noise is *black* if there does not exist any nontrivial Lévy process ξ_t such that $\xi_t - \xi_s$ is $\mathcal{F}_{s,t}$-measurable. We give an example of black noise for which the proof that it is black may be simpler than any other known examples of black noises. © 2001 Éditions scientifiques et médicales Elsevier SAS

0. Introduction

The notion of *noises* has been introduced by B. Tsirelson. In the discrete time case, it is essentially equivalent to the notion of *i.i.d. random sequences*. In the continuous time case, the most typical and important examples of noises are those generated by Gaussian white noises (i.e., increments of Wiener processes) and Poisson random measures. We call these noises as *classical*, *linearizable* or *additive*, so that, in particular, a Lévy process generates a classical noise by the Lévy–Itô decomposition theorem. An important remark was given by Tsirelson [10,11] that there exist noises in continuous time which are not isomorphic to classical noises. In particular, he showed the existence of what he calls *black noises*; roughly, a noise is black if it cannot contain any classical noise as its subnoise. It should be remarked that a notion similar to that of noises has already been introduced by Feldman [3] in the case of general time parameter sets under the name of *factored probability space* or *continuous product of probability spaces*. The study in [3] is restricted, however, to classical noises. Here, we are interested in non classical noises, black noises, in particular.

An example of black noise was first given by Tsirelson and Vershik [14] and then Tsirelson showed in [12] that the noise generated by Arratia's coalescing Brownian motion (cf. [1]) is black. The latter example is certainly more easily understandable to probabilists than the former. However, an exact proof that it is actually black still seems rather hard. The main purpose of this note is to give another example of black noise for which the proof that it is black may be much simpler. The proof in [12] that the noise generated by Arratia's coalescing Brownian motion is black has been quantitative in the sense that it is based on estimates for the conditional variance of a class of L^2-functionals of the noise. In our example, the proof will be qualitative, meaning roughly that we do not need any estimate.

To construct our example, we also introduce a coalescing stochastic flow as Arratia's, whose one-point motion, however, is a *singular diffusion* on the real line. Here, we mean, by a singular diffusion on the real line, a Feller diffusion having, as its canonical scale, the Euclidean scale and having, as its speed measure, an everywhere positive measure which, however, is singular to the Lebesgue measure. We consider a coalescing stochastic flow, formed of independent particles each obeying the law of a singular diffusion before the collision with other particles and the coalescence takes place at the collision. The existence of such a flow can be proved essentially in the same way as Arratia's case by following a general method of Harris [4]. Then we have a noise generated by this flow. We can prove that it is black by studying the space of martingales with respect to the filtration associated with the flow; its Davis–Varaiya invariant [2] is singular to that of a Brownian filtration so that it cannot contain any Wiener martingale, thereby we may conclude that the noise cannot contain any classical noise as its subnoise.

1. Noises in continuous time

As mentioned in the Introduction, the notion of noises has been introduced and studied by Tsirelson [10–13]. Before giving a formal definition, we prepare some general notions and notations. In the following, a probability space $(\Omega, \mathcal{F}, P)$ is always assumed to be complete and, when we speak of a sub σ-field of $\mathcal{F}$, it is assumed to contain all P-null sets, unless otherwise stated. The trivial σ-field, which consists of events with probability 0 or 1, is denoted simply by

$\{\Omega, \emptyset\}$. For a sub σ-field $\mathcal{G}$ of $\mathcal{F}$, we denote by $L^0(\Omega;\mathcal{G})$, or simply by $L^0(\mathcal{G})$ when Ω is well understood, the space of all $\mathcal{G}$-measurable real random variables (more precisely, the space of all the equivalence classes of $\mathcal{G}$-measurable real random variables coinciding each other P-almost surely). $L^p(\Omega;\mathcal{G})$ ($1 \leqslant p < \infty$) is the subspace of $L^0(\Omega;\mathcal{G})$ formed of all pth integrable random variables. Unless otherwise stated, the probability space $(\Omega, \mathcal{F}, P)$ is always assumed to be *separable* in the sense that the Hilbert space $L^2(\Omega;\mathcal{F})$ is a separable Hilbert space.

DEFINITION 1.1. – *Let $(\Omega, \mathcal{F}, P)$ and $(\Omega', \mathcal{F}', P')$ be two probability spaces and let $\mathcal{G}$ and $\mathcal{G}'$ be sub σ-fields of $\mathcal{F}$ and $\mathcal{F}'$, respectively. By a* morphism *π from $(\Omega, \mathcal{G})$ to $(\Omega', \mathcal{G}')$, denoted by $\pi : (\Omega, \mathcal{G}) \to (\Omega', \mathcal{G}')$, we mean a mapping*

$$\pi_* : L^0(\Omega';\mathcal{G}') \to L^0(\Omega;\mathcal{G})$$

with the following properties:

(i) *For any $X_1, \ldots, X_n \in L^0(\Omega';\mathcal{G}')$,*

$$\big[(X_1, \ldots, X_n), P'\big] \stackrel{d}{=} \big[\big(\pi_*(X_1), \ldots, \pi_*(X_n)\big), P\big].$$

(ii) *For any $X_1, \ldots, X_n \in L^0(\Omega';\mathcal{G}')$ and any Borel function $f : \mathbf{R}^n \to \mathbf{R}$,*

$$\pi_*\big[f(X_1, \ldots, X_n)\big] = f\big(\pi_*(X_1), \ldots, \pi_*(X_n)\big).$$

Remark 1.1. – Of course, what we have in mind in the above definition of morphism is a point transformation $\pi : \Omega \to \Omega'$ which is $\mathcal{G}/\mathcal{G}'$-measurable and satisfies $P' = P \circ \pi^{-1}$ on $\mathcal{G}'$, so that it induces π_* by $\pi_*(X) = X \circ \pi$, $X \in L^0(\Omega', \mathcal{G}')$. However, we would avoid mentioning a point transformation explicitly.

DEFINITION 1.2. – *By a* noise, *we mean a family $\{\mathcal{F}_{s,t}; -\infty < s \leqslant t < \infty\}$ of sub σ-fields of $\mathcal{F}$ on a probability space $(\Omega, \mathcal{F}, P)$ with the following properties*:

(1) *$\mathcal{F}_{s,u} = \mathcal{F}_{s,t} \vee \mathcal{F}_{t,u}$, and $\mathcal{F}_{s,t}$ and $\mathcal{F}_{t,u}$ are independent for every $s \leqslant t \leqslant u$, so that, in particular, $\mathcal{F}_{t,t} = \{\Omega, \emptyset\}$ for every $t \in \mathbf{R}$.*

(2) *Denoting $\mathcal{F}_{-\infty,\infty} = \bigvee_{s\leqslant t} \mathcal{F}_{s,t}$, there exists a one-parameter family $\{T_h\}_{h\in\mathbf{R}}$ of morphisms $T_h : (\Omega, \mathcal{F}_{-\infty,\infty}) \to (\Omega, \mathcal{F}_{-\infty,\infty})$ such that $(T_h)_* \circ (T_{h'})_* = (T_{h+h'})_*$, $T_0 = \mathrm{id}$ and $(T_h)_*[L^0(\mathcal{F}_{s,t})] = L^0(\mathcal{F}_{s+h,t+h})$ for every $h, h' \in \mathbf{R}$ and $s \leqslant t$. Thus, $\{T_h\}_{h\in\mathbf{R}}$ is a one-parameter group of morphisms.*

We denote the noise in this definition as $\mathbf{N} = [\{\mathcal{F}_{s,t}\}_{s\leqslant t}, \{T_h\}_{h\in\mathbf{R}}]$ *or simply as* $\mathbf{N} = \{\mathcal{F}_{s,t}\}$ *when* $\{T_h\}$ *is well understood.*

Remark 1.2. – If we define the notion of noise similarly in the case of discrete time $n \in \mathbf{Z}$, then $\{\mathcal{F}_{m,n}; m \leqslant n\}$ must be given as $\mathcal{F}_{m,n} = \sigma[\xi_{m+1}, \ldots, \xi_n]$ where $\{\xi_n\}$ is an i.i.d. random sequence. Hence, the notion of noises in the discrete time is essentially equivalent to that of i.i.d. random sequences.

Example 1.1. – Let $w = (w_t)_{-\infty<t<\infty}$ be a d-dimensional Wiener process ($1 \leqslant d \leqslant \infty$) and let $\mathcal{F}_{s,t}$, $s \leqslant t$, be the σ-field generated by $\{w_v - w_u; s \leqslant u \leqslant v \leqslant t\}$. Let T_h, $h \in \mathbf{R}$, be a morphism $T_h : (\Omega, \mathcal{F}_{-\infty,\infty}) \to (\Omega, \mathcal{F}_{-\infty,\infty})$ uniquely determined by

$$(T_h)_*\big[f(w_t - w_s)\big] = f(w_{t+h} - w_{s+h}),$$
$$\text{for any Borel function } f : \mathbf{R}^d \to \mathbf{R}.$$

Then the family $\{\mathcal{F}_{s,t}; s \leqslant t\}$ together with the one-parameter group of morphisms $\{T_h\}$ defines a noise $\mathbf{N}_w$. This noise $\mathbf{N}_w$ is called a d-dimensional *Gaussian noise* or *white noise*.

Example 1.2. – Let S be a Polish space and $n(\mathrm{d}x)$ be a σ-finite Borel measure on S. Let $p(\mathrm{d}t, \mathrm{d}x)$ be a Poisson random measure on $(-\infty, \infty) \times S$ with the mean measure $\mathrm{d}t \cdot n(\mathrm{d}x)$. Let $\mathcal{F}_{s,t}$, $s \leqslant t$, be the σ-field generated by $\{p((u, v] \times E); s \leqslant u \leqslant v \leqslant t, E \in \mathcal{B}(S)\}$ and T_h, $h \in \mathbf{R}$, be a morphism $T_h : (\Omega, \mathcal{F}_{-\infty,\infty}) \to (\Omega, \mathcal{F}_{-\infty,\infty})$ uniquely determined by

$$(T_h)_*\big[p\big((s,t] \times E\big)\big] = p\big((s+h, t+h] \times E\big), \quad s \leqslant t,\ E \in \mathcal{B}(S).$$

Then the family $\{\mathcal{F}_{s,t}; s \leqslant t\}$ together with the one-parameter group of morphisms $\{T_h\}$ defines a noise $\mathbf{N}_p$. This noise $\mathbf{N}_p$ is called a *Poissonian noise*.

More generally, we can define a noise from an independent pair of Wiener process and Poisson random measure (allowing a trivial one for each), which we call a *classical noise*, an *additive noise* or a *linearizable noise*, so that Gaussian noises and Poissonian noises are particular examples of classical noises.

DEFINITION 1.3. – *Let* $\mathbf{N} = [\{\mathcal{F}_{s,t}\}_{s\leqslant t}, \{T_h\}_{h\in\mathbf{R}}]$ *and* $\mathbf{N}' = [\{\mathcal{F}'_{s,t}\}_{s\leqslant t}, \{T'_h\}_{h\in\mathbf{R}}]$ *be two noises defined on probability spaces* Ω *and* Ω', *respectively. We say that* $\mathbf{N}'$ *is* homomorphic to $\mathbf{N}$ *if there exists a morphism* $\pi:(\Omega,\mathcal{F}_{-\infty,\infty})\to(\Omega',\mathcal{F}'_{-\infty,\infty})$ *such that*

$$\pi_*\big[L^0(\Omega',\mathcal{F}'_{s,t})\big]\subset L^0(\Omega,\mathcal{F}_{s,t}),\quad \forall s\leqslant t,\quad \textit{and}$$
$$\pi_*\circ(T'_h)_*=(T_h)_*\circ\pi_*,\quad \forall h\in\mathbf{R}.$$

If, furthermore, $\pi_*[L^0(\Omega',\mathcal{F}'_{s,t})]=L^0(\Omega,\mathcal{F}_{s,t})$ *for all* $s\leqslant t$*, then we say that* $\mathbf{N}'$ *is* isomorphic to $\mathbf{N}$.

When $\mathbf{N}'$ is isomorphic to $\mathbf{N}$, noting that π_* is always injective, we see easily that $(\pi_*)^{-1}$ defines a morphism $\pi^{-1}:(\Omega',\mathcal{F}'_{-\infty,\infty})\to(\Omega,\mathcal{F}_{-\infty,\infty})$ so that $\mathbf{N}$ is isomorphic to $\mathbf{N}'$. Thus, in this case, we may well say that $\mathbf{N}$ *and* $\mathbf{N}'$ *are isomorphic*.

DEFINITION 1.4. – *Let* $\mathbf{N}=[\{\mathcal{F}_{s,t}\}_{s\leqslant t},\{T_h\}_{h\in\mathbf{R}}]$ *and* $\mathbf{N}'=[\{\mathcal{F}'_{s,t}\}_{s\leqslant t},\{T'_h\}_{h\in\mathbf{R}}]$ *be two noises defined on a same probability space. We say that* $\mathbf{N}'$ *is a* subnoise *of* $\mathbf{N}$ *if* $\mathcal{F}'_{s,t}\subset\mathcal{F}_{s,t}$ *for all* $s\leqslant t$ *and* $(T'_h)_*$ *is the restriction of* $(T_h)_*$ *for every* $h\in\mathbf{R}$.

The following two propositions are easy to prove:

PROPOSITION 1.1. – *A noise* $\mathbf{N}'$ *is homomorphic to a noise* $\mathbf{N}$ *if and only if* $\mathbf{N}'$ *is isomorphic to a subnoise of* $\mathbf{N}$.

PROPOSITION 1.2. – (1) *A subnoise of a Gaussian noise is Gaussian. More generally, any noise homomorphic to a Gaussian noise is also Gaussian.*

(2) *Let* $\mathbf{N}$ *and* $\mathbf{N}'$ *be two Gaussian noises with dimension* d *and* d'*, respectively. Then* $\mathbf{N}'$ *is homomorphic to* $\mathbf{N}$ *if and only if* $d'\leqslant d$. $\mathbf{N}'$ *is isomorphic to* $\mathbf{N}$ *if and only if* $d'=d$.

(3) *A subnoise of a classical noise is classical. More generally, any noise homomorphic to a classical noise is also classical.*

DEFINITION 1.5. – *For a noise* $\mathbf{N}=[\{\mathcal{F}_{s,t}\}_{s\leqslant t},\{T_h\}_{h\in\mathbf{R}}]$*, define*

$$\widehat{\mathcal{F}}_{s,t}=\mathcal{F}_{-t,-s},\quad s\leqslant t,\quad \textit{and}\quad \widehat{T}_h=T_{-h},\quad h\in\mathbf{R}.$$

Then, obviously, $\widehat{\mathbf{N}}=[\{\widehat{\mathcal{F}}_{s,t}\}_{s\leqslant t},\{\widehat{T}_h\}_{h\in\mathbf{R}}]$ *is a noise and we call it the* reversed noise *of* $\mathbf{N}$.

The reversed noise of a Gaussian noise is a Gaussian noise isomorphic to it, more generally, the reversed noise of a classical noise is a classical noise isomorphic to it.

DEFINITION 1.6. – *Given a noise* $\mathbf{N}=\{\mathcal{F}_{s,t}\}_{s\leqslant t}$, *the filtration* $\mathbf{F}=\{\mathcal{F}_{0,t}\}_{t\geqslant 0}$ *is called the* filtration associated with the noise $\mathbf{N}$.

For a filtration $\mathbf{F}$, we denote, as usual, the space of locally square-integrable $\mathbf{F}$-martingales $M=(M_t)$ with $M_0=0$ by $\mathcal{M}(\mathbf{F})$ and its subspace formed of all continuous elements by $\mathcal{M}^c(\mathbf{F})$.

Remark 1.3. – Let $\mathbf{N}$ be a noise and $\mathbf{N}'$ be its subnoise. Then obviously, the filtration $\mathbf{F}'$ associated with $\mathbf{N}'$ is a subfiltration of the filtration $\mathbf{F}$ associated with $\mathbf{N}$: $\mathbf{F}'\subset\mathbf{F}$. We can easily verify that this is a martingale immersion in the sense that

$$\mathcal{M}(\mathbf{F}')\subset\mathcal{M}(\mathbf{F}).$$

DEFINITION 1.7. – *A noise* $\mathbf{N}=\{\mathcal{F}_{s,t}\}_{s\leqslant t}$ *is called* predictable *if, for the filtration* $\mathbf{F}$ *associated with* $\mathbf{N}$, *it holds that* $\mathcal{M}(\mathbf{F})=\mathcal{M}^c(\mathbf{F})$.

Gaussian noises are predictable. Poissonian noises are not predictable and, more generally, a classical noise is predictable if and only if it is Gaussian.

PROPOSITION 1.3. – *Let* $\mathbf{N}=\{\mathcal{F}_{s,t}\}_{s\leqslant t}$ *be a noise. Then there exists a maximal classical subnoise* $\mathbf{N}'$ *of* $\mathbf{N}$ *and it is unique. When* $\mathbf{N}$ *is predictable,* $\mathbf{N}'$ *is the unique maximal Gaussian subnoise of* $\mathbf{N}$.

DEFINITION 1.8. – *A noise* $\mathbf{N}'$ *in Proposition* 1.3 *is denoted by*

$$\mathbf{N}^{lin}=\left[\left\{\mathcal{F}^{lin}_{s,t}\right\}_{s\leqslant t},\{T_h\}_{h\in\mathbf{R}}\right].$$

Thus, a noise $\mathbf{N}$ is classical if and only if $\mathbf{N}=\mathbf{N}^{lin}$ and it is Gaussian if and only if it is predictable and $\mathbf{N}=\mathbf{N}^{lin}$. Tsirelson [10] called a nontrivial noise for which $\mathbf{N}^{lin}$ is trivial, i.e., $\mathcal{F}^{lin}_{s,t}=\{\Omega,\emptyset\}$ for any (= for some) $s<t$, a *black noise*.

2. A coalescing stochastic flow having a singular diffusion on R as its 1-point motion

First of all, we introduce some notations. For a positive integer n, let $\pi=(I_1,\ldots,I_k)$ be a partition of $\{1,\ldots,n\}$: $I_1\cup\cdots\cup I_k=\{1,\ldots,n\}$ (disjoint union). For two such partitions $\pi=(I_1,\ldots,I_k)$ and $\pi'=$

$(I'_1, \ldots, I'_{k'})$, we denote $\pi' \prec \pi$ if π is finer than π', that is, each I_p, $p = 1, \ldots, k$, is contained in some $I'_{p'}$, $p' = 1, \ldots, k'$. $\pi_1 := (I_1, \ldots, I_n)$ with $I_l = \{l\}$, $l = 1, \ldots, n$, is the finest partition and $\pi_0 := (I_0)$ with $I_0 = \{1, \ldots, n\}$, is the coarsest partition, so that $\pi_0 \prec \pi \prec \pi_1$ for all π.

Given a partition $\pi = (I_1, \ldots, I_k)$, let

$$\mathbf{R}^n_\pi = \bigl\{x = (x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_i = x_j \text{ if and only if } i, j \in I_p \text{ for some } p\bigr\}$$

so that we have

$$\mathbf{R}^n = \bigcup_\pi \mathbf{R}^n_\pi \text{ (disjoint union)},$$
$$\mathbf{R}^n_{\pi_0} = \bigl\{x = (x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_1 = \cdots = x_n\bigr\}$$

and

$$\mathbf{R}^n_{\pi_1} = \bigl\{x = (x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_i \neq x_j \text{ for } i \neq j\bigr\}.$$

For $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$, a partition $\pi = (I_1, \ldots, I_k)$ is uniquely determined so that $x = (x_1, \ldots, x_n) \in \mathbf{R}^n_\pi$. We let $p(x) = (x_{i_1}, \ldots, x_{i_k}) \in \mathbf{R}^k$ where $i_p \in I_p$, $p = 1, \ldots, k$, and $k = k(x)$. In other words, $k(x)$ is the number of distinct elements in $x_1, \ldots, x_n$ and $p(x) = (x_{i_1}, \ldots, x_{i_k})$ is a list of these distinct elements.

Let $W^n = \mathcal{C}([0,\infty) \to \mathbf{R}^n)$ be the Polish space of all n-dimensional continuous paths endowed with the metric of uniform convergence on bounded intervals. We say that $w = (w_1(t), \ldots, w_n(t)) \in W^n$ has the *coalescing property* if $w_{i_1}(t) = \cdots = w_{i_k}(t)$ implies $w_{i_1}(s) = \cdots = w_{i_k}(s)$ for all $s \geqslant t$. $w \in W^n$ has the coalescing property if and only if, for any $s \geqslant t$, $w(t) \in \mathbf{R}^n_\pi$ impies $w(s) \in \mathbf{R}^n_{\pi'}$ for some $\pi' \prec \pi$. The totality of $w \in W^n$ having the coalescing property is denoted by W^n_c.

DEFINITION 2.1. – (i) *For* $w \in W^n_c$ *with* $w(0) \in \mathbf{R}^n_\pi$, *set*

$$\sigma(w) = \inf\bigl\{t > 0 \mid w(t) \in \mathbf{R}^n_{\pi'} \textit{ for } \pi' \neq \pi, \pi' \prec \pi\bigr\} \tag{1}$$

and call it the first coalescing time. *Here, as usual, we understand* $\inf\emptyset = \infty$.

(ii) *Define the* m*th coalescing time* $\sigma_m(w)$, $m = 0, 1, \ldots,$ *by*

$$\begin{aligned} &\sigma_0(w) = 0, \quad \sigma_1(w) = \sigma(w) \quad \textit{and} \\ &\sigma_{m+1}(w) = \sigma_m(w) + \sigma\bigl(w^+_{\sigma_m(w)}\bigr), \end{aligned} \tag{2}$$

where, for $s \geqslant 0$, $w_s^+ \in W_c^n$ *is defined, as usual, by* $w_s^+(t) = w(s+t)$, $t \geqslant 0$.

Let $\xi = (\xi_t, P_x)_{x \in \mathbf{R}}$ be a one-dimensional diffusion process on $\mathbf{R}$ with the Feller generator

$$L = \frac{\mathrm{d}}{m(\mathrm{d}x)} \frac{\mathrm{d}}{\mathrm{d}x},$$

(cf. [6]), where the canonical scale is the Euclidean scale x and the speed measure $m(\mathrm{d}x)$ is an everywhere positive Radon measure on $\mathbf{R}$ which is singular to the Lebesgue measure. *For simplicity, we assume that* $m(\mathrm{d}x)$ *is periodic with period* 1. Then both ∞ and $-\infty$ are natural (neither exit nor entrance, cf. [6]) so that ξ is uniquely determined from L. The path $[t \mapsto \xi_t]$ is, under P_x, a W^1-valued random variable with $\xi_0 = x$, a.s. and we call it the L-diffusion starting at $x \in \mathbf{R}$.

Example 2.1. – A typical example is the case $m(\mathrm{d}x)|_{[0,1]} = \mathrm{d}F_p(x)$, where, for given $0 < p < 1$ and $p \neq \frac{1}{2}$, $F_p(x)$ on $[0,1]$ is the unique solution to the functional equation

$$F_p(x) = \begin{cases} (1-p)F_p(2x), & x \in [0, \frac{1}{2}), \\ (1-p) + pF_p(2x-1), & x \in [\frac{1}{2}, 1]. \end{cases} \tag{3}$$

It is well-known that $F_p(x)$ is continuous, strictly increasing and singular on $[0,1]$ with $F_p(0) = 0$ and $F_p(1) = 1$.

PROPOSITION 2.1. – *Let* $x \in \mathbf{R}$ *and* $\xi^x = (\xi^x(t))$ *be an* L*-diffusion starting at* x. *Then,* $M = (M(t))$ *defined by*

$$M(t) = \xi^x(t) - x, \quad t \geqslant 0,$$

is a square-integrable martingale and $E[|\xi^x(t)|^2] = \mathrm{O}(|x|^2)$ *as* $|x| \to \infty$ *for each fixed* $t > 0$. *Furthermore, the quadratic variation* $\langle M \rangle$ *is singular to the* standard time *St given by* $St(t) \equiv t$ (*cf. Section* 3 *for a precise definition of the singularity and the absolute continuity for increasing processes*).

Proof. – M is a local martingale because the canonical scale of ξ is the Euclidean scale. Since the speed measure $m(\mathrm{d}x)$ is periodic with period 1, we can easily deduce that it is square-integrable and

$E[|\xi^x(t)|^2] = \mathrm{O}(|x|^2)$ as $|x| \to \infty$ for each fixed $t > 0$. $\langle M\rangle$ is singular to the standard time *St* because the measure $m(\mathrm{d}x)$ is singular to the Lebesgue measure $\mathrm{d}x$. $\square$

For each $k = 1, 2, \ldots,$ and $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbf{R}^k_{\pi_1}$, i.e., $\mathbf{x} = (x_1, \ldots, x_k)$ with distinct $x_1, \ldots, x_k$, take k-independent L-diffusions, $\xi^{(p)} = \{\xi^p(t)\}$, $p = 1, \ldots, k$, each starting at x_p. Let

$$\zeta = \inf\{t > 0 \mid \xi^i(t) = \xi^j(t) \text{ for some } i \neq j\}$$

and let $Q^k_{\mathbf{x}}$ be a probability on W^k_c given by the law of the stopped path:

$$t \in [0, \infty) \mapsto \big(\xi^1(t \wedge \zeta), \ldots, \xi^k(t \wedge \zeta)\big).$$

Now we define, for given $\mathbf{x} \in \mathbf{R}^n$, a probability measure $\mathbf{P_x}$ on W^n_c as follows: for each $m = 0, 1, \ldots,$ the law, under the regular conditional probability $\mathbf{P_x}(**/\mathcal{F}_{\sigma_m})$, of the path

$$t \in [0, \infty) \to p\big(w\big[\big(t + \sigma_m(w)\big) \wedge \sigma_{m+1}(w)\big]\big)$$

coincides with $Q^{k[w(\sigma_m)]}_{p[w(\sigma_m)]}$. It is easy to see that such a probability $\mathbf{P_x}$ on W^n_c exists uniquely. Intuitively, $\mathbf{P_x}$ describes the motion of independent particles of L-diffusion which coalesce at the collision. Also, it is a standard argument (cf. e.g., [5, p. 370]) to conclude that $\{\mathbf{P_x}\}_{\mathbf{x} \in \mathbf{R}^n}$ defines a strong Markov process on $\mathbf{R}^n$.

DEFINITION 2.2. – *For given n and $\mathbf{x} \in \mathbf{R}^n$, a continuous process $\xi = (\xi_t)$ on $\mathbf{R}^n$ having the law $\mathbf{P_x}$ is called a* coalescing n-point L-diffusion starting at $\mathbf{x} = (x_1, \ldots, x_n)$.

If, for $n > m$, $\pi_{n,m} : \mathbf{R}^n \to \mathbf{R}^m$ is the natural projection, then it induces the projection $\pi^*_{n,m} : W^n_c \to W^m_c$ in an obvious way and, clearly, the family $\{\mathbf{P_x}\}$ is consistent in the sense that

$$\mathbf{P_x} \circ (\pi^*_{n,m})^{-1} = \mathbf{P}_{\pi_{n,m}(\mathbf{x})}, \quad \text{for } \mathbf{x} \in \mathbf{R}^n, n > m.$$

Hence, by the Kolmogorov extension theorem, we can construct a family $\{X_x = (X_x(t))\}$ of one-dimensional paths $X_x \in W^1$, indexed by $x \in \mathbf{R}$, such that, for all $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$, the law of $\mathbf{X_x} := (X_{x_1}, \ldots, X_{x_n})$ coincides with $\mathbf{P_x}$. It is obvious that, for fixed $x \leqslant y$, $X_x(t) \leqslant X_y(t)$ for all $t \geqslant 0$, a.s.

Let $\mathbf{D} \subset \mathbf{R}$ be the set of all dyadic rationals and set

$$X_x^*(t) = \inf_{y \in \mathbf{D}, y \geqslant x} X_y(t), \quad t \geqslant 0.$$

We can deduce, easily from the Feller property of the L-diffusion, that $X_x^*(t)$ is continuous in t and $X_x^*(t) = X_x(t)$ for all $t \geqslant 0$, a.s. for each $x \in \mathbf{R}$.

Let Φ be the set of all non-decreasing right-continuous functions $\varphi : x \in \mathbf{R} \mapsto \varphi(x) \in \mathbf{R}$ with the metric defined by $\rho(\varphi, \psi) = \sum_{n=1}^{\infty} 2^{-n} \times (\rho_n(\varphi, \psi) \wedge 1)$ where

$$\rho_n(\varphi, \psi) = \inf\bigl\{\varepsilon > 0 \mid \varphi(x - \varepsilon) - \varepsilon \leqslant \psi(x) \leqslant \varphi(x + \varepsilon) + \varepsilon \\ \text{for all } x \in [-n, n]\bigr\}.$$

Then Φ is a Polish space and the composite $(\varphi, \psi) \in \Phi \times \Phi \mapsto \psi \circ \varphi \in \Phi$ is $(\mathcal{B}(\Phi) \times \mathcal{B}(\Phi))/\mathcal{B}(\Phi)$-measurable, where $\mathcal{B}(\Phi)$ is the topological σ-field of Φ. Also, we see that, for each $t \in [0, \infty)$, $\mathbf{X}_t : x \in \mathbf{R} \mapsto X_x^*(t) \in \mathbf{R}$ defines a $(\Phi, \mathcal{B}(\Phi))$-valued random variable. Let $\mathcal{P}_t$ be the law of $\mathbf{X}_t$. Then, as in [4], we have (denoting by id the function $\varphi(x) \equiv x$)

$$\mathcal{P}_s * \mathcal{P}_t = \mathcal{P}_{s+t}, \quad s,\ t \geqslant 0, \quad \mathcal{P}_0 = \delta_{\mathrm{id}}, \quad \text{and} \\ \mathcal{P}_t \to \delta_{\mathrm{id}} \text{ weakly as } t \downarrow 0,$$

where, for two probabilities $\mathcal{P}$ and $\mathcal{Q}$ on $(\Phi, \mathcal{B}(\Phi))$, the convolution $\mathcal{P} * \mathcal{Q}$ is defined, as usual, to be the law of $\psi \circ \varphi$ under the product measure $\mathcal{P}(\mathrm{d}\varphi) \times \mathcal{Q}(\mathrm{d}\psi)$. We call such a family $\{\mathcal{P}_t\}$ a one-parameter convolution semigroup of probabilities on $(\Phi, \mathcal{B}(\Phi))$.

Now we recall a general definition of stochastic flows (cf. [7,4]); let S be a topological space and $\mathcal{T}$ be a class of transformations on S containing the identity (denoted by id) and forming a semigroup under the composite. We assume that a suitable metric topology is given on $\mathcal{T}$ such that, if $\mathcal{B}(\mathcal{T})$ is the topological σ-field of $\mathcal{T}$, $(\mathcal{T}, \mathcal{B}(\mathcal{T}))$ is a standard Borel space (cf. [9, Chap. V, 2]). We assume further that the composite $(\tau_1, \tau_2) \in \mathcal{T} \times \mathcal{T} \mapsto \tau_1 \circ \tau_2 \in \mathcal{T}$ is $(\mathcal{B}(\mathcal{T}) \times \mathcal{B}(\mathcal{T}))/\mathcal{B}(\mathcal{T})$-measurable. Assume further that the composite $(\tau_1, \tau_2) \mapsto \tau_1 \circ \tau_2$ is continuous at (τ_1, τ_2) if either τ_1 or τ_2 is equal to id. (These conditions are satisfied in the case $\mathcal{T} = \Phi$.)

DEFINITION 2.3. – *By a $\mathcal{T}$-stochastic flow, we mean a family of $\mathcal{T}$-valued random variables $\tau = \{\tau_{s,t}; -\infty < s \leqslant t < \infty\}$ having the following properties*:

(1) (*the flow property*) $\tau_{s,u} = \tau_{t,u} \circ \tau_{s,t}$ *and* $\tau_{t,t} = \mathrm{id}$, *a.s. for every* $s \leqslant t \leqslant u$,

(2) (*the independent increment property*) *for any sequence* $t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n$, *$\mathcal{T}$-valued random variables* τ_{t_{k-1},t_k}, $k = 1, \dots, n$, *are independent,*

(3) (*the stationarity*) *for any* $h > 0$, $\tau_{s,t} \stackrel{d}{=} \tau_{s+h,t+h}$,

(4) (*the stochastic continuity*) $\tau_{0,h} \to \mathrm{id}$ *in probability as* $h \downarrow 0$.

LEMMA 2.1. – *For given convolution semigroup $\{\mu_t\}_{t \geqslant 0}$ of probabilities on $(\mathcal{T}, \mathcal{B}(\mathcal{T}))$, there exists a $\mathcal{T}$-stochastic flow $\tau = \{\tau_{s,t}; -\infty < s \leqslant t < \infty\}$ such that the law of $\tau_{s,t}$ coincides with μ_{t-s}. Furthermore, the law of such a flow is unique.*

This lemma, maybe well-known, can be proved by a standard application of the Kolmogorov extension theorem. For completeness, we give an outline of the proof.

Proof of Lemma 2.1. – Let $I = \{\lambda = (s,t); s \leqslant t\}$ and, for $\lambda_1, \dots, \lambda_n \in I$, define a Borel probability $Q_{\lambda_1,\dots,\lambda_n}$ on the n-fold product $\mathcal{T}^n$ of $\mathcal{T}$ as follows; let $\lambda_i = (s_i, t_i)$ and let $\{u_0 < u_1 < \cdots < u_l\}$ be the set $\bigcup_{i=1}^n \{s_i, t_i\}$ arranged in the order of elements. Take mutually independent $\mathcal{T}$-valued random variables $\xi_1, \dots, \xi_l$ such that ξ_k is distributed by $\mu_{u_k - u_{k-1}}$ ($k = 1, \dots, l$). Define $\mathcal{T}$-valued random variable η_{λ_i} ($i = 1, \dots, n$), by

$$\eta_{\lambda_i} = \begin{cases} \mathrm{id} & \text{when } s_i = t_i, \\ \xi_{k+m} \circ \cdots \circ \xi_{k+1} & \text{when } s_i = u_k < \cdots < u_{k+m} = t_i. \end{cases}$$

Finally, define $Q_{\lambda_1,\dots,\lambda_n}$ to be the law on $\mathcal{T}^n$ of $(\eta_{\lambda_1}, \dots, \eta_{\lambda_n})$.

We can easily verify that the family $\{Q_{\lambda_1,\dots,\lambda_n}\}$ satisfies the consistency condition so that, by the Kolmogorov extension theorem for random variables taking values in a standard Borel space, we can construct a family $\{\tau_\lambda; \lambda \in I\}$ of $\mathcal{T}$-valued random variables such that the law of $(\tau_{\lambda_1}, \dots, \tau_{\lambda_n})$ coincides with $Q_{\lambda_1,\dots,\lambda_n}$. Then, $\tau_{s,t} = \tau_\lambda$, $\lambda = (s,t)$, is what we want. Note that the construction is possible on a separable probability space because we have $\tau_{s,t} = \lim_{m\to\infty} \lim_{n\to\infty} \tau_{s_m,t_n}$ in probability where s_m and t_n are dyadic rationals such that $s_m \uparrow s$ and $t_n \downarrow t$.

The uniqueness in law of $\{\tau_{s,t}\}_{s\leqslant t}$ is obvious. $\square$

Applying this lemma to the convolution semigroup $\{\mathcal{P}_t\}$ defined above in the case $(\mathcal{T}, \mathcal{B}(\mathcal{T})) = (\Phi, \mathcal{B}(\Phi))$, we can construct, uniquely in law, a stochastic flow $\mathbf{X} = \{X_{s,t};\ s \leqslant t\}$ on $\mathbf{R}$.

DEFINITION 2.4. – *This stochastic flow* $\mathbf{X}$ *is called the* coalescing L-stochastic flow.

The following proposition is almost obvious.

PROPOSITION 2.2. – *For* $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$ *and* $s \in \mathbf{R}$, *the process* $\mathbf{X}(t) = (X_{s,s+t}(x_1), \ldots, X_{s,s+t}(x_n))$ *has the law* $\mathbf{P}_{\mathbf{x}}$, *that is,* $\mathbf{X}(t)$ *is a coalescing* n*-point* L*-diffusion starting at* $\mathbf{x}$. *In other words, the* n*-point motion of the flow* $\mathbf{X}$ *is a coalescing* n*-point* L*-diffusion.*

3. The noise generated by the coalescing L-stochastic flow is black

Generally, if a $\mathcal{T}$-stochastic flow $\tau = \{\tau_{s,t}; -\infty < s \leqslant t < \infty\}$ is given on a complete probability space $(\Omega, \mathcal{F}, P)$, we define, for each $s \leqslant t$, a sub σ-field $\mathcal{F}_{s,t}$ of $\mathcal{F}$ as the smallest σ-field containing all P-null sets, with respect to which, all $\mathcal{T}$-valued random variables $\tau_{u,v}$, $s \leqslant u \leqslant v \leqslant t$, are measurable. We set further, for $-\infty < t < \infty$,

$$\mathcal{F}_{-\infty,t} = \bigvee_{-\infty<u\leqslant t} \mathcal{F}_{u,t}, \quad \mathcal{F}_{t,\infty} = \bigvee_{t\leqslant u<\infty} \mathcal{F}_{t,u} \quad \text{and}$$

$$\mathcal{F}_{-\infty,\infty} = \bigvee_{-\infty<u\leqslant v<\infty} \mathcal{F}_{u,v}.$$

Then, for each $h \in \mathbf{R}$ and $s \leqslant t$, there exists a unique morphism $T_h : (\Omega, \mathcal{F}_{s+h,t+h}) \to (\Omega, \mathcal{F}_{s,t})$ such that

$$(T_h)_*\big(\tau_{s,t}(x)\big) = \tau_{s+h,t+h}(x), \quad \text{for all } x \in \mathbf{R}.$$

We can easily deduce, from the properties (1), (2) and (3) in Definition 2.3 of $\mathcal{T}$-flow, the properties (1) and (2) in Definition 1.2 of the noise, so that we can conclude that $\mathbf{N} = [\{\mathcal{F}_{s,t}\}_{s\leqslant t}, \{T_h\}_{h\in\mathbf{R}}]$ defines a noise.

Let $\mathbf{X} = \{X_{s,t}; -\infty < s \leqslant t < \infty\}$ be the coalescing L-stochastic flow of Definition 2.4. Then it defines a noise. We denote this noise by $\mathbf{N}^{\mathbf{X}} = [\{\mathcal{F}^{\mathbf{X}}_{s,t}\}_{s\leqslant t}, \{T_h\}_{h\in\mathbf{R}}]$ and call it *the noise generated by the coalescing* L*-stochastic flow.*

THEOREM 3.1. – *The noise* $\mathbf{N}^{\mathbf{X}} = [\{\mathcal{F}^{\mathbf{X}}_{s,t}\}_{s\leqslant t}, \{T_h\}_{h\in\mathbf{R}}]$ *generated by the coalescing* L*-stochastic flow is predictable. Furthermore, it is black, that is, it cannot contain any nontrivial classical noise as its subnoise.*

The rest of this paper is devoted to the proof of this theorem.

Let $\mathbf{N}^{\mathbf{X}} = [\{\mathcal{F}^{\mathbf{X}}_{s,t}\}_{s\leqslant t}, \{T_h\}_{h\in\mathbf{R}}]$ be the noise generated by a coalescing L-stochastic flow $\mathbf{X} = \{X_{s,t}\}$ realized on a probability space $(\Omega, \mathcal{F}, P)$ and let $\mathbf{F}^{\mathbf{X}} = \{\mathcal{F}^{\mathbf{X}}_{0,t}\}_{t\geqslant 0}$ be the filtration associated with the flow $\mathbf{N}^{\mathbf{X}}$. We denote by $\mathcal{M}_2(\mathbf{F}^{\mathbf{X}})$ the space of all square-integrable $\mathbf{F}^{\mathbf{X}}$-martingales $M = (M(t))$ with $M(0) = 0$, a.s.

PROPOSITION 3.1. – *For a fixed* $s \geqslant 0$ *and* $x \in \mathbf{R}$*, define* $M^{(s,x)} = (M^{(s,x)}(t))$ *by*

$$M^{(s,x)}(t) = \begin{cases} 0, & t \leqslant s, \\ X_{s,t}(x) - x, & t \geqslant s. \end{cases} \tag{4}$$

Then, $M^{(s,x)} \in \mathcal{M}_2(\mathbf{F}^{\mathbf{X}})$ *and* $\langle M^{(s,x)}\rangle$ *is singular to the standard time* $St = (St(t))$ *a.s., where* $St(t) \equiv t$; *that is, in the notation given below,* $\langle M^{(s,x)}\rangle \perp St$.

More generally, if $\eta \in L^2(\mathcal{F}^{\mathbf{X}}_{0,s})$*, then* $M^{(s,\eta)} \in \mathcal{M}_2(\mathbf{F}^{\mathbf{X}})$ *and* $\langle M^{(s,\eta)}\rangle \perp St$*. Here,* $M^{(s,\eta)} = (M^{(s,\eta)}(t))$ *is defined by*

$$M^{(s,\eta)}(t) = \begin{cases} 0, & t \leqslant s, \\ X_{s,t}(\eta) - \eta, & t \geqslant s. \end{cases} \tag{5}$$

Proof. – By Proposition 2.2, $[t \mapsto X_{s,s+t}(x)]$ is an L-diffusion starting at x and, by Proposition 2.1, $M^{(s,x)}$ is a square-integrable martingale with $\langle M^{(s,x)}\rangle \perp St$. Hence, it only remains to show that $M^{(s,x)}$ is a martingale with respect to a larger filtration $\mathbf{F}^{\mathbf{X}}$. This is easily deduced from the formula

$$M^{(s,x)}(v) - M^{(s,x)}(u) = X_{u,v}\big(X_{s,u}(x)\big) - X_{s,u}(x), \quad s \leqslant u \leqslant v,$$

combined with the independence of $X_{u,v}$ and $\mathcal{F}^{\mathbf{X}}_{0,u}$. $\square$

Denoting as above the set of dyadic rationals by $\mathbf{D}$ and the subset of nonnegative elements by $\mathbf{D}^+$, we define a countable family $\mathcal{N}$ of $\mathcal{M}_2(\mathbf{F}^{\mathbf{X}})$ by

$$\mathcal{N} = \big\{M^{(s,x)} \mid s \in \mathbf{D}^+,\ x \in \mathbf{D}\big\}. \tag{6}$$

Clearly, $\mathbf{F}^{\mathbf{X}}$ is the smallest filtration with respect to which all $M \in \mathcal{N}$ are adapted. In particular, $\mathbf{F}^{\mathbf{X}}$ is separable in the sense that $L^2(\mathcal{F}^{\mathbf{X}}_{0,\infty})$ is a separable Hilbert space.

In the following, we mainly follow Davis–Varaiya [2] for notions and notations concerning the space $\mathcal{M}_2(\mathbf{F}^{\mathbf{X}})$ of martingales (cf. also [8]). In particular, we denote, for continuous $\mathbf{F}^{\mathbf{X}}$-adapted increasing processes A and B, $A \perp B$ when A and B are singular to each other, a.s.; $A \prec B$ when A is absolutely continuous with respect to B, a.s. More precisely, introducing a measure m_A on the $\mathbf{F}^{\mathbf{X}}$-predictable σ-field $\mathcal{P}$ on $[0,\infty) \times \Omega$ by

$$m_A(F) = \sum_{n=1}^{\infty} 2^{-n} E\left[1 \wedge \int_0^n \mathbf{1}_F(s,\cdot)\,\mathrm{d}A(s)\right], \quad F \in \mathcal{P},$$

$A \perp B$ if and only if m_A and m_B are singular to each other; $A \prec B$ if and only if m_A is absolutely continuous with respect to m_B.

$\mathcal{M}_2(\mathbf{F}^{\mathbf{X}})$ is a real Fréchet space with a system of Hilbertian seminorms

$$\|M\|_t = E\left[M(t)^2\right] = E\left[\langle M\rangle(t)\right], \quad t \geqslant 0.$$

For $M \in \mathcal{M}_2(\mathbf{F}^{\mathbf{X}})$ and an $\mathbf{F}^{\mathbf{X}}$-predictable process $\Phi = (\Phi(s))$ satisfying

$$E\left[\int_0^t \left|\Phi(s)\right|^2 \mathrm{d}\langle M\rangle(s)\right] < \infty \quad \text{for all } t > 0, \tag{7}$$

the *stochastic integral* $\Phi \cdot M \in \mathcal{M}_2(\mathbf{F}^{\mathbf{X}})$, $\Phi \cdot M(t) = \int_0^t \Phi(s)\,\mathrm{d}M(s)$, is defined. A linear subspace $\mathcal{L}$ of $\mathcal{M}_2(\mathbf{F}^{\mathbf{X}})$ is called *stable* if it satisfies that $\Phi \cdot M \in \mathcal{L}$ for any $M \in \mathcal{L}$ and an $\mathbf{F}^{\mathbf{X}}$-predictable process $\Phi = (\Phi(s))$ satisfying (7).

PROPOSITION 3.2. – *The closed and stable subspace $\mathcal{L}(\mathcal{N})$ generated by $\mathcal{N}$ (i.e., the smallest closed and stable subspace containing $\mathcal{N}$) coincides with $\mathcal{M}_2(\mathbf{F}^{\mathbf{X}})$:*

$$\mathcal{M}_2\left(\mathbf{F}^{\mathbf{X}}\right) = \mathcal{L}(\mathcal{N}). \tag{8}$$

From this proposition, Theorem 3.1 follows at once. Indeed, $\mathcal{M}_2(\mathbf{F}^{\mathbf{X}}) = \mathcal{M}_2^c(\mathbf{F}^{\mathbf{X}})$ by (8) so that the noise $\mathbf{N}^{\mathbf{X}}$ is predictable. Also,

$$\mathcal{L}(\mathcal{N}) = \mathcal{L}(N_1, N_2, \ldots, N_l), \quad 1 \leqslant l \leqslant \infty,\ N_i \in \mathcal{L}(\mathcal{N}),$$

and we may choose N_i to satisfy the conditions (i) $\sim$ (iii) of Theorem 1 in [2]. In particular, $\langle M\rangle \prec \langle N_1\rangle$ for every $M \in \mathcal{L}(\mathcal{N})$, and hence, for every $M \in \mathcal{M}_2(\mathbf{F}^{\mathbf{X}})$ by (8). Since N_1 is a countable sum of stochastic integrals by elements in $\mathcal{N}$, it is obvious that $\langle N_1\rangle \perp St$ where St is the standard time; $St(t) \equiv t$. Now, suppose that the noise $\mathbf{N}^{\mathbf{X}}$ contains a Gaussian white noise as its subnoise. Then, $\mathcal{M}_2(\mathbf{F}^{\mathbf{X}})$ contains at least one Wiener martingale (i.e., a continuous martingale M with $\langle M\rangle = St$) so that we must have $St \prec \langle N_1\rangle$. This obviously contradicts the fact that $\langle N_1\rangle \perp St$.

Proof of Proposition 3.2. – We first state a martingale representation result for the n-point coalescing L-diffusion of Definition 2.2.

LEMMA 3.1. – *Let $\xi = (\xi(t))$, $\xi(t) = (\xi^1(t), \ldots, \xi^n(t))$, be an n-point coalescing L-diffusion starting at $\mathbf{x} = (x_1, \ldots, x_n)$ with the natural filtration $\mathbf{F}^{\xi}$. Set $M^i(t) = \xi^i(t) - x_i$, $i = 1, \ldots, n$. Then, $M^i \in \mathcal{M}_2(\mathbf{F}^{\xi})$ and every $M \in \mathcal{M}_2(\mathbf{F}^{\xi})$ has a representation*

$$M = \sum_{i=1}^{n} \Psi_i \cdot M^i, \quad \textit{that is}, \quad M(t) = \sum_{i=1}^{n} \int_0^t \Psi_i(s)\, \mathrm{d}M^i(s),$$

where $\Psi_i = (\Psi_i(s))$ is $\mathbf{F}^{\xi}$-predictable and satisfies

$$E\left[\int_0^t \left|\Psi_i(s)\right|^2 \mathrm{d}\left\langle M^i\right\rangle(s)\right] < \infty \quad \textit{for all } t > 0.$$

This lemma is a consequence of a general martingale representation result for Hunt processes in [8] which can be applied to the n-point coalescing L-diffusion.

COROLLARY 3.1. – *Let $f(x) = f(x_1, \ldots, x_n)$ be a bounded Borel function on $\mathbf{R}^n$. Then, for $t_i \in [0, \infty)$, $i = 1, \ldots, n$, with $t := \max\{t_1, \ldots, t_n\}$,*

$$f\big(\xi^1(t_1),\dots,\xi^n(t_n)\big) = E\big[f\big(\xi^1(t_1),\dots,\xi^n(t_n)\big)\big] + \sum_{i=1}^{n}\int_0^t \Psi^i(s)\,\mathrm{d}M^i(s), \tag{9}$$

where $\Psi_i = (\Psi_i(s))$ *is* $\mathbf{F}^\xi$*-predictable and satisfies* $E[\int_0^t |\Psi_i(s)|^2\,\mathrm{d}\langle M^i\rangle(s)] < \infty$.

Now we return to the coalescing flow $\mathbf{X}$ and the noise $\mathbf{N}^{\mathbf{X}}$ generated by it. For each fixed $s \geqslant 0$ and $x \in \mathbf{R}$, $M^{(s,x)}$ in (4) can be approximated in the space $\mathcal{M}_2(\mathbf{F}^{\mathbf{X}})$ by elements in $\mathcal{N}$ as closely as we want, so that $M^{(s,x)} \in \mathcal{L}(\mathcal{N})$. More generally, we have $M^{(s,\eta)} \in \mathcal{L}(\mathcal{N})$ for $\eta \in L^2(\mathcal{F}^{\mathbf{X}}_{0,s})$. Indeed, noting the right-continuity in x of $X_{s,t}(x)$, we have

$$M^{(s,\eta)} = \lim_{m\to\infty}\sum_{k=-\infty}^{\infty} \mathbf{1}_{[(k-1)2^{-m}\leqslant \eta < k2^{-m}]}\cdot M^{(s,k2^{-m})}.$$

By Proposition 2.2 and Corollary 3.1, we can easily deduce the following:

COROLLARY 3.2. – *Let* $s \geqslant 0$ *be fixed and let* $f(\omega,x) = f(\omega,x_1,\dots,x_n)$ *be a bounded* $\mathcal{F}^{\mathbf{X}}_{0,s}\times\mathcal{B}(\mathbf{R}^n)$*-measurable function on* $\Omega\times\mathbf{R}^n$*. Let* $\eta_1,\dots,\eta_n \in L^2(\mathcal{F}^{\mathbf{X}}_{0,s})$*. We set, for given* $t_i > s$, $i = 1,\dots,n$,

$$F(\omega) = f\big(\omega, X_{s,t_1}(\eta_1),\dots,X_{s,t_n}(\eta_n)\big).$$

Then, putting $t := \max\{t_1,\dots,t_n\}$*, we have*

$$F(\omega) = E\big[F(\omega)\mid \mathcal{F}^{\mathbf{X}}_{0,s}\big] + \sum_{i=1}^{n}\int_s^t \Psi_i(u)\,\mathrm{d}M^{(s,\eta_i)}(u), \tag{10}$$

where $\Psi_i = (\Psi_i(u))$ *is* $\mathbf{F}^{\mathbf{X}}$*-predictable with* $E[\int_s^t |\Psi_i(u)|^2\,\mathrm{d}\langle M^{(s,\eta_i)}\rangle(u)] < \infty$.

Now we can complete the proof of Proposition 3.2: It is sufficient to show that, for every bounded Borel function $f(y) = f(y_1,\dots,y_n)$ on

$\mathbf{R}^n$, $x_i \in \mathbf{R}$, and $0 \leqslant s_i < t_i$, $i = 1, \ldots, n$, the following holds:

$$F(\omega) = E(F) + \sum_{j=1}^{m} \int_{u_j}^{t} \Psi_j(u)\, \mathrm{d}M^{(u_j,\eta_j)}(u) \tag{11}$$

for some m, $u_j \geqslant 0$, $\eta_j \in L^2(\mathcal{F}^{\mathbf{X}}_{0,u_j})$, $j = 1, \ldots, m$, and $t > u_j$ for all j, where

$$F(\omega) = f\big(X_{s_1,t_1}(x_1), \ldots, X_{s_n,t_n}(x_n)\big) \tag{12}$$

and $\Psi_j = (\Psi_j(u))$ is $\mathbf{F}^{\mathbf{X}}$-predictable and satisfies $E[\int_{u_j}^{t} |\Psi_j(u)|^2 \times \mathrm{d}\langle M^{(u_j,\eta_j)}\rangle(u)] < \infty$. Here, $t = \max\{t_1, \ldots, t_n\}$.

For the proof of (11), we may assume, without loss of generality, that

$$0 \leqslant s_1 \leqslant \cdots \leqslant s_{k-1} < s_k = s_{k+1} = \cdots = s_n.$$

If $0 \leqslant j \leqslant k-1$ and $t_j > s_k$, then $X_{s_j,t_j}(x_j) = X_{s_k,t_j}(X_{s_j,s_k}(x_j)) := X_{s_k,t_j}(\eta_j)$ where $\eta_j = X_{s_j,s_k}(x_j) \in L^2(\mathcal{F}^{\mathbf{X}}_{0,s_k})$. Thus, $F(\omega)$ in (12) can be expressed in the form

$$F(\omega) = \widetilde{f}\big(\omega, X_{s_k,t'_1}(\eta'_1), \ldots, X_{s_k,t'_l}(\eta'_l)\big) \tag{13}$$

for some l, where $\widetilde{f}(\omega, y) = \widetilde{f}(\omega, y_1, \ldots, y_l)$ is a bounded $\mathcal{F}^{\mathbf{X}}_{0,s_k} \times \mathcal{B}(\mathbf{R}^l)$-measurable function on $\Omega \times \mathbf{R}^l$ and $\eta'_i \in L^2(\mathcal{F}^{\mathbf{X}}_{0,s_k})$, $i = 1, \ldots, l$. By Corollary 3.2,

$$F(\omega) = E\big[F(\omega) \mid \mathcal{F}^{\mathbf{X}}_{0,s_k}\big] + \sum_{i=1}^{l} \int_{s_k}^{t} \Psi'_i(u)\, \mathrm{d}M^{(s_k,\eta'_i)}(u). \tag{14}$$

By the independence of $\{X_{u,v};\, s_k \leqslant u \leqslant v\}$ and $\mathcal{F}^{\mathbf{X}}_{0,s_k}$, we have

$$\begin{aligned} E\big[F(\omega) \mid \mathcal{F}^{\mathbf{X}}_{0,s_k}\big] = \int_{\Lambda} \cdots \int_{\Lambda} & \widetilde{f}(\omega, \varphi_1 \circ \eta'_1, \ldots, \varphi_l \circ \eta'_l) \\ & \times P(X_{s_k,t'_1} \in \mathrm{d}\varphi_1, \ldots, X_{s_k,t'_l} \in \mathrm{d}\varphi_l). \end{aligned}$$

If we compare the expressions (12) and (13) of $F(\omega)$, we see that

$$\widetilde{f}(\omega, y_1, \ldots, y_l) = g\big(X_{\alpha_1,\beta_1}(a_1), \ldots, X_{\alpha_q,\beta_q}(a_q), y_1, \ldots, y_l\big)$$

by some Borel function $g(z_1, \ldots, z_q, y_1, \ldots, y_l)$ and some $\alpha_i \leqslant \beta_i \leqslant s_k$, $a_i \in \mathbf{R}$. Also, η'_i is either $x_{j'}$ or $\eta_j = X_{s_j,s_k}(x_j)$. Hence,

$$E\left[F(\omega) \mid \mathcal{F}^{\mathbf{X}}_{0,s_k}\right] = h\left(X_{s'_1,t'_1}(x'_1), \ldots, X_{s'_p,t'_p}(x'_p)\right)$$

for some p, some bounded Borel function $h(y) = h(y_1, \ldots, y_p)$ on $\mathbf{R}^p$ and some $s'_i \leqslant t'_i$ with $t'_i \leqslant s_k$ for all $i = 1, \ldots, p$.

Now, we can repeat the same argument for $G(\omega) := E[F(\omega) \mid \mathcal{F}^{\mathbf{X}}_{0,s_k}]$ to obtain a similar expression as (14) in which $F(\omega)$, t and s_k are replaced by $G(\omega)$, s_k and s_{k-1}, respectively. Continuing this process successively, we finally obtain the expression (11) for $F(\omega)$ given by (12). $\square$

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